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A Characterization of QF - rings

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It is well known that T. Nakayama found quasi - Frobenius rings (QF-rings) as a generalization of group algebras [3]. Nowadays we know many characterizations of QF-rings. One of them is that R is self-injective as a right R-module. Other one is that every indecomsable and projective right ideal e_1R contains the unique minimal right ideal r_1 and if $e_1R \neq e_2R$, $r_1 \neq r_2$ and the above statments are valid for left ideals, where the e_i is a primitive idempotent.

In this note we shall combine the two weakened conditions, We always assume that R is a left and right artinian ring with identity and every R-module is a unitary right R-module.

Let M be an R-module. We consider a diagram for a minimal right ideal I of R:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "f"'] & R \arrow[dl, dashed, "h"] \\ & M & \end{tikzcd}$$

If there exists $h \in \mathrm{Hom}_R(R,M)$ with $f = hi$ for any f and i, we say M is right mini-injective. We can define similarly left mini-injective. If R is right mini-injective as a right R-module, we say R is right self mini-injective. We note that right self mini-injective ring is not left self mini-injetive in general.

We shall show

Theorem 1. Let R be a right and left artinian ring. Then R is QF-ring if and only if R is right and left self mini-injective.

Theorem 2. Let R be an algebra over a field K of finite dimension. Then R is a QF-algebra if and only if R is a right self mini-injective algebra.

Remark. Theorem 2 is not valid for artinian rings.

Proofs of Theorems.

We shall publish the results in this note in [1] and [2] and so we give here a sketch of the proofs.

In this note, we consider only artinian rings and so from now on we understand a ring R is always right artinian. We denote the Jacobson radical by $J = J(R)$. We put $\overline{R} = R/J$. Let

$$R = \sum_{i=1}^{n}\sum_{j=1}^{\rho(i)} \oplus\, e_{ij}R$$

be the standard decomposition, namely the e_{ij} is a primitive idempotent and $e_{ij}R \simeq e_{i1}R$, $e_{i1}R \not\simeq e_{j1}R$ if $i \neq j$. If the socle $S(e_{i1}R)$ of $e_{i1}R$ is simple for each i, then we say R is right QF-2.

Proposition 1. Let R be as above. If R is right self mini-injective, then

1) If $e_1R \not\simeq e_2R$, each minimal submodule in e_1R is not isomorphic to any minimal one in e_2R.

2) $S(e_1R) = e_1J^k$ and every minimal submodule in e_1R is isomorphic each other.

3) $r(J) \supseteq l(J)$.

where the e_i is a primitive idempotent, $r(J) = \{x \in R \mid Jx = 0\}$ and $l(J) = \{x \in R \mid xJ = 0\}$.

Proof. 1) is clear from the definition.

2) We take a minimal right ideal I in $e_1J^k \neq 0$ ($e_1J^{k+1} = 0$). Using the definition and 1), we can show $S(e_1R) = e_1J^k$.

3) Let x be in a minimal right ideal in e_1R. Then we can show $Jx \subseteq \sum_{i=1}^{n} e_iJx = 0$ from 1) and 2), where $1 = \sum_{i=1}^{n} e_i$.

Proof of Theorem 2.

First we shall show that R is right QF-2. Let I_1 be a minimal right ideal in e_1R. We assume $I_1 \simeq \overline{e_2'R}$. Thenwe can show from the definition and Proposition 1 that there is a monomorphism of $\mathrm{End}_R(\overline{e_2'R})$ to $\mathrm{End}_R(\overline{e_1R})$. Hence, $[\overline{e_1Re_1} : K] = [\overline{e_2'Re_2'} : K]$. Next take a minimal right ideal I_2 in e_2R. Repeating this argument, we have a chain $\{e_1, e_2', \ldots\}$ of primitive idempotents. Then $e_1 \simeq et'$ for some t by Proposition 1. Hence,

$[\overline{e_1Re_1}:K] = [\overline{e_2'Re_2'}:K]$ implies that I_1 is a unique minimal right ideal in e_1R. Accordingly e_iR is uniform and so an injective encelope $E(e_iR)$ of e_iR is indecomposable. Therefore, $E(e_iR) \simeq \mathrm{Hom}_K(Re_{\pi(i)}, K)$. Since $S(e_iR) \not\simeq S(e_jR)$ if $e_iR \not\simeq e_jR$ by Proposition 1, $[R;K] = \sum_{i=1}^n [Re_{\pi(i)} :K] = \sum_{i=1}^n [E(e_iR):K] = [E(R):K]$.

Proof of Theorem 1.

Let xR be a minimal right ideal in e_1R and $xR \simeq \overline{e_2'R}$. Since $Jx = 0$ by Proposition 1 and R is left self mini-injective, for any element b in $\overline{e_1Re_1}$, there exists a in $\overline{e_2'Re_2'}$ such that $bx = xa$ as above. Hence, $xR = S(e_1R)$ and R is left and right QF-2.

References

[1] M. Harada, On self mini-injective rings, to appear.

[2] _________, A characterization of QF-rings, to appear.

[3] T. Nakayama, On Frobenius algebras II, Ann. of Math. 40(1941), 1 - 21.